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Real Hypersurfaces with Killing Shape Operator in the Complex Quadric

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Abstract. We introduce the notion of Killing shape operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. The Killing shape operator condition implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of Hopf real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with Killing shape operator.

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Keywords. Killing shape operator, $\mathfrak{A}$-isotropic, $\mathfrak{A}$-principal, Kähler structure, Complex conjugation, Complex quadric.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [15–17]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$, some classifications of real hypersurfaces related to commuting Ricci tensor were investigated by Kimura [9], and Pérez and Suh [11,12] respectively. The classification problems of real hypersurfaces in the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ with certain geometric conditions were mainly discussed in Jeong et al. [2], Jeong et al. [3,4], Suh [15–17], where the classification of *contact hypersurfaces*, *parallel Ricci*

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tensor, *harmonic curvature* and *structure Jacobi operator* of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied. Moreover, in [17] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can consider the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Klein [5,6,8] and Smyth [14]). The complex quadric can also be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [10]). Accordingly, the complex quadric admits two important geometric structures, an S^1-bundle $\mathfrak{A}$ of real structures and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$ for every $A\in\mathfrak{A}$. Then for $m\geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [5,7] and Reckziegel [13]). This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m.

Moreover, the derivative of the complex conjugation A on Q^m is defined by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M, where q denotes a certain 1-form defined on M.

When the shape operator S of M in Q^m satisfies $(\nabla_X S)Y = (\nabla_Y S)X$ for any X, Y tangent to M in Q^m, we say that the shape operator is of *Codazzi type*. In [18] we gave a non-existence result on such real hypersurfaces as follows:

Theorem A. *There do not exist any real hypersurfaces in complex quadric Q^m, $m\geq 3$, with shape operator of Codazzi type.*

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A) := \mathrm{Eig}(A, 1)$, then W is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is called $\mathfrak{A}$-isotropic.

When we consider a hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties, the unit normal vector field N of M in Q^m is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [18,19]). In the first case, where N is $\mathfrak{A}$-isotropic, we have shown in Suh [18] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [19]).

The shape operator S of M in Q^m is said to be *Killing* if the operator S satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0$$

for any $X, Y \in T_z M$, $z \in M$. The equation is equivalent to $(\nabla_X S)X = 0$ for any $X \in T_z M$, $z \in M$, because of linearization. The geometric meaning of this condition is as follows:

Consider a geodesic γ with initial conditions $\gamma(0) = z$ and $\dot{\gamma}(0) = X$. Then the transformed vector field $S\dot{\gamma}$ is Levi-Civita *parallel* along the geodesic γ of the vector field X (see Blair [1] and Tachibana [21]).

When we consider a real hypersurface in the complex quadric Q^m with Killing shape operator, we can assert

Main Theorem 1. *Let M be a Hopf real hypersurface in Q^m, $m \geq 3$, with Killing shape operator. Then the unit normal vector field N is $\mathfrak{A}$-isotropic.*

Then, motivated by such result, we give a complete classification for real hypersurfaces in the complex quadric Q^m with Killing shape operator as follows:

Main Theorem 2. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with Killing shape operator. Then M has 4 distinct constant principal curvatures given by*

$$\alpha \neq 0, \;\; \beta = \gamma = 0, \;\; \lambda = \frac{(\alpha^2+1) + \sqrt{(\alpha^2+1)^2 + 2\alpha^2}}{2\alpha}, \;\; \textit{and}$$
$$\mu = \frac{(\alpha^2+1) - \sqrt{(\alpha^2+1)^2 + 2\alpha^2}}{2\alpha}$$

whose corresponding principal curvature spaces are

$$T_\alpha = [\xi], T_\beta = [AN], T_\gamma = [A\xi], \phi(T_\lambda) = T_\mu, \;\; \textit{and dim}\; T_\lambda = \textit{dim}\; T_\mu = m-2.$$

Remark 1.1. Usually, the Killing shape operator is a generalization of the parallel shape operator S of M in Q^m , that is, $\nabla_X S = 0$ for any tangent vector field X on M. The parallelism of shape operator has a geometrical meaning that every eigen space of the shape operator S is parallel in any direction on M in Q^m. Then naturally, by the equation of Codazzi in Section 3, we can prove easily that there do not exist any Hopf real hypersurface in Q^m, $m \geq 3$, with parallel shape operator (see [18]).

2. The Complex Quadric

For more background to this section, we refer to [5,10,13,18–20]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini–Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini–Study metric $\bar{g}$ is defined by

$\bar{g}(X,Y) = \Phi(JX,Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z_0,\ldots,z_j,\ldots,z_{m+1}]\in\mathbb{C}P^{m+1}|z_j\neq 0\}$, where the function f_j is given by $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z_k}{z_j}$ for $j,k = 0,\ldots,m+1$. Naturally, the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J,g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0,\ldots,0,1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason, we will assume $m \geq 3$ from now on.

In addition, the complex projective space $\mathbb{C}P^{m+1}$ is defined using the Hopf fibration

$$\pi : S^{2m+3}\rightarrow\mathbb{C}P^{m+1}, \quad z\rightarrow[z],$$

which is a Riemannian submersion. Then, we can consider the following diagram for the complex quadric Q^m:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\tilde{i}} & S^{2m+3}\subset\mathbb{C}^{m+2} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{m+2}$, which is given by

$$\tilde{Q} = \{x+iy\in\mathbb{C}^{m+2}|g(x,x) = g(y,y) = \frac{1}{2} \text{ and } g(x,y) = 0\},$$

where $g(x,y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1,\ldots,x_{m+2})$ and $y = (y_1,\ldots,y_{m+2})$ $\in\mathbb{R}^{m+2}$. Then, the tangent space is decomposed as $T_zS^{2m+3} = H_z\oplus F_z$ and $T_z\tilde{Q} = H_z(Q)\oplus F_z(Q)$ at $z = x+iy\in\tilde{Q}$ respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^\perp$ and $H_z(Q) = (\mathbb{C}z\oplus\mathbb{C}\bar{z})^\perp$, and $F_z = F_z(Q) = \mathbb{R}iz$ is the tangent space to the fiber $S^1{\cdot}z$ of π through the point z. Here $H_z(Q)$ is a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x+iy\in\tilde{Q}$ it can be described as

$$H_z = \{u+iv\in\mathbb{C}^{m+2}| \quad g(x,u)+g(y,v) = 0, \quad g(x,v) = g(y,u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z| \quad g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$, and $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)}Q$ respectively. This gives that at the point $\pi(z) = [z]$, the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see Reckziegel [13]).

Then let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}} w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$. That is, the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}} w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda \bar{w} \\ &= \lambda A_{\bar{z}} \lambda \bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda \bar{\lambda} \bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. Thus, the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}} X = X$ and $A_{\bar{z}} JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically, this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$, or equivalently, is a complex conjugation on $T_{[z]}Q^m$. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $[z] \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at $[z]$. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at $[z]$ of such a reflection is a conjugation on $T_{[z]}Q^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_{[z]}Q^m$ corresponds to the family of real forms S^m of Q^m containing $[z]$, and the subspaces $V(A) \subset T_{[z]}Q^m$ correspond to the tangent spaces $T_{[z]}S^m$ of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

For every unit tangent vector $W \in T_{[z]}Q^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. When $W = X$ for $X{\in}V(A)$, $t = 0$, X is contained in all 2-flats $\mathbb{R}X + \mathbb{R}JZ$ with $Z{\in}V(A)$ orthogonal to X. So the tangent vector X is said to be singular. When $W = (X + JY)/\sqrt{2}$ for $t = \frac{\pi}{4}$, it is also a singular tangent vector, which belongs to all 2-flats $\mathbb{R}\tilde{X} + \mathbb{R}J\tilde{Y}$, where

$$\tilde{X} = \frac{1+\lambda}{2}X + \frac{1-\lambda}{2}JY$$

and

$$\tilde{Y} = -\frac{1-\lambda}{2}JX + \frac{1+\lambda}{2}Y$$

for some $\lambda{\in}S^1$. If $0 < t < \frac{\pi}{4}$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$.

3. Some General Equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that the Reeb vector field of M is given by $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ, which is defined by $\phi X = JX - \eta(X)N$, restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z{\in}M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \quad \text{for all } A \in \mathfrak{A}_z\}.$$

Then we introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1. [18] *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the Reeb vector field $\xi = -JN$ satisfies the following

$$S\xi = \alpha\xi,$$

where S denotes the shape operator of the real hypersurface M for a smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider the transformed JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then, we now consider the equation of Codazzi

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ & \quad = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ & \quad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ & \quad = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & \qquad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ & \qquad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

From the above equations we obtain

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [13]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 = 2g(S\phi SX, Y) &- \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

4. Killing Shape Operator and a Key Lemma

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X &- g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY \\ &+ g(SY, Z)SX - g(SX, Z)SY \end{aligned}$$

for any $X, Y, Z \in T_zM$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m$, $z \in M$, $\rho(X) = g(AX, N)$, where BX and $\rho(X)N$, respectively, denote the tangential and normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned} AN = AJ\xi &= -JA\xi = -J(B\xi + \rho(\xi)N) \\ &= -(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

The shape operator S of M in Q^m is said to be *Killing* if it satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0. \tag{4.1}$$

for any $X, Y \in T_zM$, $z \in M$.

From (4.1), together with the equation of Codazzi (3.1), it follows that

$$\begin{aligned} 2g((\nabla_X S)Y, Z) = \eta(X)g(\phi Y, Z) &- \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ &+ g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &+ g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{4.2}$$

Since we have assumed the real hypersurface M in Q^m is *Hopf*, then $S\xi = \alpha\xi$. This gives

$$(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX.$$

From this, let us put $Y=\xi$ in (4.2) and use $g(A\xi,N)=0$. We see that

$$2g((X\alpha)\xi+\alpha\phi SX-S\phi SX,Z)=-g(\phi X,Z)+g(X,AN)g(A\xi,Z) \tag{4.3}$$

$$+g(X,A\xi)g(JA\xi,Z)-g(\xi,A\xi)g(JAX,Z). \tag{4.4}$$

Here, let us take $X=\xi$ in (4.3) and also use $g(\xi,AN)=0$. We have

$$2(\xi\alpha)\eta(Z)=g(\xi,A\xi)g(JA\xi,Z)-g(\xi,A\xi)g(JA\xi,Z)=0.$$

From this we get $\xi\alpha=0$. Then the derivative $Y\alpha$ in Sect. 3 becomes

$$Y\alpha=2g(Y,AN)g(\xi,A\xi).$$

From this, together with (4.3), it follows that

$$2g(2g(X,AN)g(\xi,A\xi)\xi+\alpha\phi SX-S\phi SX,Z)=-g(\phi X,Z)+g(X,AN)g(A\xi,Z)$$
$$+g(X,A\xi)g(JA\xi,Z)-g(\xi,A\xi)g(JAX,Z). \tag{4.5}$$

Then by putting $Z=\xi$ into (4.3), we have

$$\begin{aligned}4g(X,AN)g(\xi,A\xi)&=g(X,AN)g(A\xi,\xi)+g(X,A\xi)g(JA\xi,\xi)\\&\quad-g(\xi,A\xi)g(JAX,\xi)\\&=2g(X,AN)g(A\xi,\xi).\end{aligned} \tag{4.6}$$

Since $g(A\xi,N)=0$, (4.6) gives

$$g(A\xi,\xi)g(AN,X)=0.$$

Then we have $g(A\xi,\xi)=0$ or $(AN)^{\mathrm{T}}=0$, where $(AN)^{\mathrm{T}}$ denotes the tangential part of the vector AN.

We will use the result of this discussion to prove the following

Lemma 4.1. *Let M be a Hopf real hypersurface in Q^m, $m\geq 3$, with Killing shape operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. In the above discussion, let us consider the first case $g(A\xi,\xi)=0$. Then it implies that

$$0=g(A\xi,\xi)=g(AJN,JN)=-g(JAN,JN)=-g(AN,N).$$

If we insert $N=\cos tZ_1+\sin tJZ_2$ for $Z_1,Z_2\in V(A)$ into the above equation, we have $\cos^2 t-\sin^2 t=0$. By section 2, we have $t=\frac{\pi}{4}$, that is, $N=\frac{1}{\sqrt{2}}(X+JY)$ for some $X,Y\in V(A)$. So the unit normal N is $\mathfrak{A}$-isotropic.

Next we consider the case that $(AN)^{\mathrm{T}}=0$. Then $AN=(AN)^{\mathrm{T}}+g(AN,N)N=g(AN,N)N$. So it follows that

$$N=A^2N=g(AN,N)AN=g^2(AN,N)N.$$

So $g(AN,N)=\pm 1$ gives that $AN=\pm N$. That is, the unit normal N is $\mathfrak{A}$-principal. $\square$

Due to Lemma 4.1, the classification of Hopf real hypersurfaces with Killing shape operator in Q^m splits into two cases, depending on the unit normal N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. We will study these two cases in the following two sections. In Sect. 5, we will obtain the classification of Hopf real hypersurfaces in Q^m with Killing shape operator and $\mathfrak{A}$-isotropic unit normal vector field and in Sect. 6 a non-existence of Hopf real hypersurfaces with Killing shape operator and $\mathfrak{A}$-principal vector field will be proved.

5. Proof of Main Theorem with $\mathfrak{A}$-Isotropic Unit Normal

In this section, let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. The normal vector field N can be written

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ \ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2),\ \text{and}\ JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (3.3) and the anti-commuting property $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \ \ \text{and} \ \ g(AN, N) = 0.$$

Now (4.3) gives the following for any $X, Z \in T_zM$, $z \in M$

$$\begin{aligned} 2g(\alpha\phi SX - S\phi SX, Z) &= -g(\phi X, Z) + g(X, AN)g(A\xi, Z) + g(X, A\xi)g(JA\xi, Z) \\ &= -g(\phi X, Z) + g(X, AN)g(A\xi, Z) - g(X, A\xi)g(AN, Z). \end{aligned} \tag{5.1}$$

Since $A\xi, AN \in T_xM$, $x \in M$, it implies

$$2(\alpha\phi SX - S\phi X) = -\phi X + g(X, AN)A\xi - g(X, A\xi)AN. \tag{5.2}$$

On the other hand, from the formula (5.6) of Suh [19] for a Hopf real hypersurface M with $\mathfrak{A}$-isotropic unit normal N

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \tag{5.3}$$

Then by virtue of (5.2) and (5.3), we have

$$-2S\phi SX = \alpha S\phi X - 3\alpha\phi SX. \tag{5.4}$$

We know that the tangent space T_zM, $z \in M$ is decomposed as follows:

$$T_zM = [\xi] \oplus [A\xi, AN] \oplus \mathcal{Q},$$

where $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^\perp = \text{Span}[A\xi, AN]$.

Lemma 5.1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field. Then*

$$SAN = 0, \quad \text{and} \quad SA\xi = 0.$$

Proof. Let us denote by $\mathcal{C}\ominus\mathcal{Q} = \mathcal{Q}^{\perp} = \text{Span}[A\xi, AN]$. Since N is $\mathfrak{A}$-isotropic, $g(AN, N) = 0$ and $g(A\xi, \xi) = 0$. By differentiating $g(AN, N) = 0$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(AN), N) + g(AN, \bar{\nabla}_X N) \\ &= g(q(X)JAN - ASX, N) - g(AN, SX) \\ &= -2g(ASX, N) \\ &= -2g(X, SAN) \end{aligned}$$

Then $SAN = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have the following formula

$$\begin{aligned} 0 &= g(\bar{\nabla}_X(A\xi), N) + g(A\xi, \bar{\nabla}_X N) \\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ &= -2g(SA\xi, X) \end{aligned}$$

for any $X\in T_zM$, $z\in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

By Lemma 5.1 we know that the distribution $\mathcal{Q}^{\perp}$ for a Hopf real hypersurface M in Q^m is invariant by the shape operator S, so the distribution $\mathcal{Q}$ is also S-invariant. From this fact, we may consider a principal curvature vector $X\in\mathcal{Q}$ such that $SX = \lambda X$. Then (5.4) gives

$$(2\lambda + \alpha)S\phi X = 3\alpha\lambda\phi X.$$

If $2\lambda+\alpha = 0$ holds, then this equation would imply $3\alpha\lambda\phi X = 0$, and therefore, as $\alpha\neq 0$ and $\lambda = -\frac{\alpha}{2}\neq 0$, we would have $\phi X = 0$. But this is impossible for $X\in\mathcal{Q}$. Thus we have $2\lambda + \alpha\neq 0$ and hence we obtain

$$S\phi X = \frac{3\alpha\lambda}{2\lambda + \alpha}\phi X. \tag{5.5}$$

For $X\in\mathcal{Q}$, we know that $g(X, AN) = g(X, A\xi) = 0$. So (5.3) gives the following

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \tag{5.6}$$

That is, for $X\in\mathcal{Q}$ such that $SX = \lambda X$ the formula (5.6) yields

$$2\lambda S\phi X = \alpha S\phi X + (\alpha\lambda + 2)\phi X. \tag{5.7}$$

If $\alpha = 2\lambda$, we should have $2(\lambda^2 + 1)\phi X = 0$, which is impossible. Then we get $S\phi X = \mu\phi X$ with

$$\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}. \tag{5.8}$$

Then (5.5) and (5.8) give

$$\frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X = \frac{3\alpha\lambda}{2\lambda + \alpha}\phi X.$$

From this, any principal curvatures λ and μ of the distribution $\mathcal{Q}$ satisfy the following quadratic equation

$$2\alpha\lambda^2 - 2(\alpha^2+1)\lambda - \alpha = 0. \tag{5.9}$$

The solutions become the following constant principal curvatures given by

$$\lambda, \mu = \frac{(\alpha^2+1)\pm\sqrt{(\alpha^2+1)^2+2\alpha^2}}{2\alpha}, \tag{5.10}$$

because the Reeb function α is constant for $\mathfrak{A}$-isotropic unit normal N (see [18]). Here we note that the Reeb function α can not vanish. If the function α identically vanishes, then (5.9) gives $\lambda = 0$. From this, together with (5.7), we have $\phi X = 0$, which implies a contradiction.

From this, together with Lemma 5.1, the expression of the shape operator becomes the following

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu \end{bmatrix}$$

where the principal curvatures λ and μ are given by (5.10) with multiplicities $m-2$ respectively. If both had different multiplicities, then $(\alpha^2+1)^2+2\alpha^2 = 0$, which is impossible.

Summing up the above discussions, we give the following

Theorem 5.2. *Let M be a real hypersurface in the complex quadric Q^m with $\mathfrak{A}$-isotropic unit normal vector field. Then M has 4 distinct constant principal curvatures given by*

$$\alpha\neq 0,\ \ \beta=\gamma=0,\ \ \lambda = \frac{(\alpha^2+1)+\sqrt{(\alpha^2+1)^2+2\alpha^2}}{2\alpha},\ \textit{and}$$
$$\mu = \frac{(\alpha^2+1)-\sqrt{(\alpha^2+1)^2+2\alpha^2}}{2\alpha}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi], T_\beta = [AN], T_\gamma = [A\xi], \phi(T_\lambda) = T_\mu,\ \textit{and dim}\ T_\lambda = \textit{dim}\ T_\mu = m-2.$$

6. Proof of Main Theorem with $\mathfrak{A}$-Principal Normal Vector Field

In this section, let us consider a real hypersurface M in Q^m with Killing shape operator for the case that the unit normal N is $\mathfrak{A}$-principal. Choose

$A\in\mathfrak{A}$ so that $N\in V(A)$ holds. Then the Killing shape operator condition gives that

$$2g(\{\alpha\phi SX - S\phi SX\}, Z) = -g(\phi X, Z) + g(\phi AX, Z),$$

where we have used $g(\xi, A\xi) = -1$ and $JAX = \phi AX + \eta(AX)N$. It follows that

$$2(\alpha\phi SX - S\phi SX) = -\phi X + \phi AX. \tag{6.1}$$

Since the unit normal vector field N is $\mathfrak{A}$-principal, $A\xi = -\xi$. Then differentiating this and using Gauss equation, we get

$$\nabla_X(A\xi) = \bar{\nabla}_X(A\xi) - g(SX, A\xi)N = -q(X)N + \alpha\eta(X)N, \tag{6.2}$$

where q denotes a certain 1-form defined on M as in the introduction. From this, together with $\nabla_X(A\xi) = -\nabla_X\xi = -\phi SX$, we have

$$\phi X = \phi AX.$$

This gives

$$AX = X - 2\eta(X)\xi.$$

Then we have

$$\begin{aligned} TrA &= g(AN, N) + \sum_{i=1}^{2m-1} g(Ae_i, e_i) \\ &= \sum_{i=1}^{2m-1} g(e_i - 2\eta(e_i)\xi, e_i) \\ &= 2(m-1). \end{aligned} \tag{6.3}$$

But $TrA = 0$, because $T_zQ^m = V(A)\oplus JV(A)$, where $V(A) = \{X\in T_zQ^m | AX = X\}$ and $JV(A) = \{X\in T_zQ^m | AX = -X\}$. This gives us a contradiction. So we obtain the

Theorem 6.1. *There does not exist any Hopf real hypersurface in the complex quadric Q^m with Killing shape operator if the unit normal vector field is $\mathfrak{A}$-principal.*

Summing up all of discussions including Sects. 4 and 5, by Lemma 4.1, Theorems 5.2 and 6.1, we give a complete proof of our Main Theorem 1 in the introduction.

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References

[1] Blair, D.E.: Almost contact manifolds with Killing structure tensors. Pac. J. Math. **39**, 285–292 (1971)

[2] Jeong, I., Kim, H.J., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator. Publ. Math. Debr. **76**, 203–218 (2010)

[3] Jeong, I., Machado, C.J.G., Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator. Int. J. Math. **22**, 655–673 (2011)

[4] Jeong, I., Machado, C.J.G., Pérez, J.D., Suh, Y.J.: $\mathcal{D}$-parallelism of normal and structure Jacobi operators for hypersurfaces in complex two-plane Grassmannians. Ann. Mat. Pura Appl. **193**, 591–608 (2014)

[5] Klein, S.: Totally geodesic submanifolds in the complex quadric. Differ. Geom. Appl. **26**, 79–96 (2008)

[6] Klein, S.: Totally geodesic submanifolds of the complex quadric and the quaternionic 2-Grassmannians. Trans. Am. Math. Soc. **361**, 4927–4967 (2009)

[7] Klein, S.: Reconstructing the geometric structure of a Riemannian symmetric space from its Satake diagram. Geom. Dedic. **138**, 25–50 (2009)

[8] Klein, S.: Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank 2. Osaka J. Math. **47**, 1077–1157 (2010)

[9] Kimura, M.: Real hypersurfaces and complex submanifolds in complex projective space. Trans. Am. Math. Soc. **296**, 137–149 (1986)

[10] Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II (Wiley classics library edition). Wiley, New York (1996)

[11] Pérez, J.D., Suh, Y.J.: Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. Differ. Geom. Appl. **7**, 211–217 (1997)

[12] Pérez, J.D., Suh, Y.J.: Certain conditions on the Ricci tensor of real hypersurfaces in quaternionic projective space. Acta Math. Hung. **91**, 343–356 (2001)

[13] Reckziegel, H.: On the geometry of the complex quadric. In: Geometry and Topology of Submanifolds VIII. World Scientific Publishing, Brussels/Nordfjordeid, River Edge, pp. 302–315 (1995)

[14] Smyth, B.: Differential geometry of complex hypersurfaces. Ann. Math. **85**, 246–266 (1967)

[15] Suh, Y.J.: Real hypersurfaces of type B in complex two-plane Grassmannians. Monatsh. Math. **147**, 337–355 (2006)

[16] Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. R. Soc. Edinb. A. **142**, 1309–1324 (2012)

[17] Suh, Y.J.: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. **50**, 645–659 (2013)

[18] Suh, Y.J.: Real hypersurfaces in the complex quadric with Reeb parallel shape operator. Int. J. Math. **25**, 1450059 (2014)

[19] Suh, Y.J.: Real hypersurfaces in the complex quadric with parallel Ricci tensor. Adv. Math. **281**, 886–905 (2015)

[20] Suh, Y.J.: Real hypersurfaces in the complex quadric with harmonic curvature. J. Math. Pures Appl. **106**, 393–410 (2016)

[21] Tachibana, S.: On Killing tensors in a Riemannian space. Tohoku Math. J. **20**, 257–264 (1968)

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